Pressure-Strain and Pressure-Scalar Gradient Correlations in Variable-Density Turbulent Flows

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Pressure-containing correlations appearing in balance equations for turbulent stresses and scalar fluxes are studied theoretically. In particular, correlations between the fluctuating pressure and fluctuating strain rate or scalar gradient fluctuations are considered. New forms that differ somewhat from others in the literature are proposed to model these terms. The consequences of the new models are shown to be extreme in flows with highly variable density, such as flames, but the models collapse to accepted models in constant-density flows. Where possible, the models are compared against experiments, and the results are encouraging.

Nomenclature

a_{ij} = anisotropy tensor
$\tilde{a}^{mi}_{\ell j}$ = fourth-order tensor in pressure-strain correlation
$\tilde{b}^{m}_{ij}$ = third-order tensor in pressure-scalar gradient correlation
C_{ij} = velocity-pressure gradient correlation scalar
c = scalar
$c_2, c_{\Phi 1}, c_{\Phi 2}$ = constants in correlations
D_k = scalar-pressure gradient correlation
D_{ij} = tensor appearing in pressure-strain correlation
f = correlation function in second-order isotropic tensor
G = Green's function in second-order isotropic tensor
g = correlation function in second-order isotropic tensor
K = correlation function in first-order isotropic tensor
k = Favre-averaged turbulent kinetic energy
ℓ = integral length scale of the turbulence
P_{ij} = turbulence production tensor
P = P_{ii}
p = pressure
r = distance between x and y
t = time
V = volume of all space
v_i = velocity in ith direction
x_i, y_i = Cartesian coordinates in ith direction
ξ_i = $x_i - y_i$
ρ = density
Φ = pressure-strain correlation
φ = pressure-scalar gradient correlation

Superscripts

$(\tilde{\ })$ = Favre-averaged quantity
$(\bar{\ })$ = conventional time or ensemble average
$(\ ')$ = conventional fluctuation
$(\ '')$ = Favre fluctuation

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I. Introduction

In the analytical treatment of turbulent reacting flows by methods of second-order closure, several correlations appear involving both the mean and fluctuating pressure.[1] Modeling of these correlations has been extremely difficult for researchers because experimental evidence on which to base models is scant. In fact, such evidence is practically nonexistent. Some progress has been made in constant-density flows, and indirect evidence shows that the proposed models appear reasonable.[2,3] However, extension of these models to the variable-density case is not a trivial matter.

In its most primitive form, pressure enters the stress balance equations in the form

$$C_{ij} = \overline{v''_i \frac{\partial p}{\partial x_j}} \tag{1}$$

and the scalar flux balance in the form

$$D_k = \overline{c'' \frac{\partial p}{\partial x_k}} \tag{2}$$

Common practice is to expand Eqs. (1) and (2) exactly into the forms

$$C_{ij} = \overline{v''_i} \frac{\partial \bar{p}}{\partial x_j} + \frac{\partial}{\partial x_j} \overline{v''_i p'} - \overline{p' \frac{\partial v''_i}{\partial x_j}} \tag{3}$$

$$D_k = \overline{c''} \frac{\partial \bar{p}}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{p' c''} - \overline{p' \frac{\partial c''}{\partial x_j}} \tag{4}$$

The first term on the right side of Eqs. (3) and (4) is generally considered not to be troublesome and will not be treated in this paper. The second terms have been treated by many methods in the literature. These second terms are known to be important in both constant- and variable-density flows, and the overbarred quantities are known as the pressure-velocity and pressure-scalar correlations. Investigation of these correlations is the subject of much current research, but these correlations will not be pursued further in this paper. Of interest here are the last terms on the right sides of Eqs. (3) and (4), called the pressure-strain and pressure-scalar gradient correlations.

In constant-density flows, a successful model for the pressure-strain correlation has been developed[2] using a prior suggestion by Rotta[4] and some arguments based on incompressibility, symmetry, weak variation of mean quantities, and near homogeneity of the turbulence. In this model, two kinds of terms enter having the forms

$$\overline{\bar{\rho} v_\ell' v_m'} \quad \text{or} \quad \overline{\bar{\rho} v_i' v_j'} \frac{\partial \bar{v}_k}{\partial x_\ell}$$

where two indices of the latter are always contracted to form a second-order tensor. In the absence of any experimental evidence, Jones[1] suggested that in variable-density flows the model, with some variations due to compressibility, be retained but that the given forms be changed to

$$\bar{\rho} \widetilde{v_\ell'' v_m''} \quad \text{or} \quad \bar{\rho} \widetilde{v_i'' v_j''} \frac{\partial \tilde{v}_k}{\partial x_\ell}$$

The model using these forms has some recent experimental support[5] in diffusion flames, but it must be recognized that other equally valid forms could have been chosen. For example,

$$\bar{\rho} \widetilde{v_\ell' v_m'} \quad \text{or} \quad \widetilde{v_i' v_j'} \frac{\partial}{\partial x_\ell} \bar{\rho} \tilde{v}_k$$

could have been chosen with the same attribute that they reduce to the proper constant-density forms in the limit of constant density.

Following the same argument for the pressure-strain model, a model for the pressure-scalar gradient correlation was developed[3] which contains the forms

$$\overline{\bar{\rho} v_i' c'} \frac{\partial \bar{v}_j}{\partial x_k}, \qquad \overline{\bar{\rho} v_j' c'}, \qquad \text{or} \quad \overline{\bar{\rho} v_i' v_j'}\, \overline{v_k' c'}$$

with two indices of the first and third forms contracted to form a vector. Again Jones[1] suggests for the variable-density case that these forms be replaced by

$$\bar{\rho} \widetilde{v_i'' c''} \frac{\partial \tilde{v}_j}{\partial x_k}, \qquad \bar{\rho} \widetilde{v_j'' c''}, \qquad \text{or} \quad \bar{\rho} \widetilde{v_i'' v_j''}\, \widetilde{v_k'' c''}$$

However, it must be recognized that this suggestion is not unique, just as with the pressure-strain correlation.

It is the purpose here to revisit the analytical basis for the pressure-strain and pressure-scalar gradient correlations. On the basis of some prior arguments and some assumed properties of the density field, it will be shown that a more natural set of forms appears for the given models when extended to the variable-density case. New models will, therefore, be proposed and will be compared against available experimental evidence.

II. Theory

Pressure Fluctuations

Consider first the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

Splitting density into its Reynolds average plus fluctuation, splitting velocity into the Favre average plus fluctuation, and subtracting the Reynolds-averaged equation $\partial(\bar{\rho}\tilde{v}_j)/\partial x_j = 0$ results in

$$\frac{\partial v_i''}{\partial x_i} = -v_i'' \frac{\partial}{\partial x_i} \ell n \bar{\rho} \tag{5}$$

This gives the departure of the velocity fluctuation from incompressibility. It is the product of two terms that may conceptually be made small; i.e., in the case of near-constant density and weak turbulence, the velocity fluctuations will be nearly divergence-free. Since the original constant-density models for the pressure-strain and pressure-scalar gradients use incompressibility in the derivation, it is essential here to see how large the effect is. Equation (5) will be used later.

The momentum equation, neglecting viscous effects, is

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i}$$

Using the same operations as with the continuity equation and taking the divergence of the result yields, after tedious calculation and use of the continuity equation,

$$-\frac{\partial^2 p'}{\partial x_j \partial x_j} = -\frac{\partial^2 \rho'}{\partial t^2} + 2\frac{\partial^2}{\partial x_i \partial x_j}(\rho' \tilde{v}_j v_i'') + \frac{\partial^2}{\partial x_i \partial x_j}(\rho v_i'' v_j'' - \overline{\rho v_i'' v_j''})$$
$$+ 2\frac{\partial^2}{\partial x_i \partial x_j}(\bar{\rho} \tilde{v}_j v_i'') \tag{6}$$

Except for the first two terms explicitly involving the density fluctuation, Eq. (6) resembles that of Chou,[6] which was used in development of the original pressure-strain and pressure-scalar gradient correlations.[2,3] The only differences are that velocity appears in Favre form and the density is variable.

Consider now the limit that $\rho'/\bar{\rho} \ll 1$ and drop the first two terms of Eq. (6). This is not to say that ρ is a constant. Consider also that wall effects are absent (jets, wakes, etc.) so that the flow is in unbounded space. A solution to the Poisson equation for pressure is

$$p'(x,t) = \int \mathrm{d}V(y) G(x,y) \left\{ \frac{\partial^2}{\partial y_\ell \partial y_m}(\rho v_\ell'' v_m'' - \overline{\rho v_\ell'' v_m''}) \right.$$
$$\left. + 2\frac{\partial^2}{\partial y_\ell \partial y_m}(\bar{\rho} \tilde{v}_\ell v_m'') \right\} \tag{7}$$

where $G = 1/(4\pi|x-y|)$ is the free-space Green's function.

Equation (7) may now be used to form the correlations of interest. They are

$$\Phi_{ij} \equiv \overline{p' \frac{\partial v_i''}{\partial x_j}} \equiv \Phi_{ij,1} + \Phi_{ij,2}$$

$$\Phi_{ij,1} = \int \mathrm{d}V(y)\, G \overline{\frac{\partial v_i''}{\partial x_j} \frac{\partial^2}{\partial y_\ell \partial y_m}(\rho v_\ell'' v_m'' - \overline{\rho v_i'' v_m''})}$$

$$\Phi_{ij,2} = 2\int \mathrm{d}V(y)\, G \overline{\frac{\partial v_i''}{\partial x_j} \frac{\partial^2}{\partial y_\ell \partial y_m}(\bar{\rho} \tilde{v}_\ell v_m'')} \tag{8}$$

$$\varphi_i \equiv \overline{p' \frac{\partial c''}{\partial x_i}} \equiv \varphi_{i,1} + \varphi_{i,2}$$

$$\varphi_{i,1} = \int \mathrm{d}V\, G \overline{\frac{\partial c''}{\partial x_i} \frac{\partial^2}{\partial y_\ell \partial y_m}(\rho v_\ell'' v_m'' - \overline{\rho v_i'' v_m''})}$$

$$\varphi_{i,2} = 2\int \mathrm{d}V\, G \overline{\frac{\partial c''}{\partial x_i} \frac{\partial^2}{\partial y_\ell \partial y_m}(\bar{\rho} \tilde{v}_\ell v_m'')} \tag{9}$$

The problem is now of modeling the integrals in Eqs. (8) and (9). In analogy with constant density treatments, $\Phi_{ij,1}$ and $\varphi_{j,1}$ are called the "return to isotropy" terms and $\Phi_{ij,2}$ and $\varphi_{i,2}$ are called the "rapid" terms.

Modeling the Correlations

In view of some recent success in a variable-density flow with high anisotropy,[5] Jones'[1] suggestion for the "return to isotropy" terms will be accepted here. Also, there has been found no real justification to alter it. The same will not be true for the "rapid" terms. Accordingly, the "return to isotropy" terms will be merely written here, after noting that in stress transport equations, the pressure-strain correlation always appears as $\Phi_{ij} + \Phi_{ji}$.

$$\Phi_{ij,1} + \Phi_{ji,1} = -c_1\bar{\rho}\frac{\varepsilon}{k}a_{ij}; \qquad c_1 \approx 1.5$$

$$\varphi_{i,1} = -\bar{\rho}\frac{\varepsilon}{k}(c_{\varphi 1}\widetilde{v_i''c''} + c_{\varphi 2}a_{ij}\widetilde{v_j''c''}); \qquad c_{\varphi 1} \approx 4 \quad c_{\varphi 2} \approx ?$$

where a_{ij} is the anisotropy tensor given by

$$a_{ij} = \widetilde{v_i''v_j''} - \frac{2}{3}\delta_{ij}k$$

No numerical value for $c_{\varphi 2}$ is given here since it is experimentally uncertain.[7]

Equations (8) may now be split into two terms since

$$\frac{\partial^2}{\partial y_\ell \partial y_m}\bar{\rho}\tilde{v}_\ell v_m'' = \frac{\partial}{\partial y_m}(\bar{\rho}\tilde{v}_\ell)\frac{\partial v_m''}{\partial y_\ell} + \bar{\rho}\tilde{v}_\ell\frac{\partial^2 v_m''}{\partial y_m \partial y_\ell}$$

Then

$$\Phi_{ij,2,I} = 2\int \mathrm{d}V(y)\, G\frac{\partial}{\partial y_m}(\bar{\rho}\tilde{v}_\ell)\frac{\partial^2}{\partial x_j \partial y_\ell}\overline{v_m''(y)v_i''(x)} \quad (10)$$

$$\Phi_{ij,2,II} = 2\int \mathrm{d}V(y)\, G\bar{\rho}\tilde{v}_\ell\frac{\partial^3}{\partial x_j \partial y_m \partial y_\ell}\overline{v_i''(x)v_m''(y)} \quad (11)$$

Now assume the turbulence field to be nearly isotropic so that[8]

$$\overline{v_i''(x)v_m''(y)} \approx \overline{v_i''v_m''(y)}\left[\frac{f(r)-g(r)}{r^2}\xi_i\xi_m + g(r)\delta_{im}\right]$$

$$r = |y - x|, \qquad \xi_i = x_i - y_i$$

If the correlation drops to zero within a distance small compared with that of significant variations in $\bar{\rho}\tilde{v}_\ell$, Eq. (11) will vanish since the triple derivative in the integrand will be odd in the ξ_j. This set of affairs is assumed approximately true so that at least $\Phi_{ij,2,II} \ll \Phi_{ij,2,I}$.

The situation is now very close to that of Ref. 2. Making a near-homogeneity assumption, noting

$$\left.\frac{\partial}{\partial y_i}\right)_{x_i} = -\left.\left(\frac{\partial}{\partial \xi_i} + \frac{\partial}{\partial x_i}\right)\right)_{y_i}$$

$$\left.\frac{\partial}{\partial x_i}\right)_{y_i} = \left.\left(\frac{\partial}{\partial \xi_i} + \frac{\partial}{\partial x_i}\right)\right)_{\xi_i}$$

and assuming the second derivative of $\bar{\rho}\tilde{v}_\ell$ is weak,

$$\Phi_{ij,2,I} = \tilde{a}_{\ell j}^{mi}\frac{\partial}{\partial x_m}(\bar{\rho}\tilde{v}_\ell) \approx \Phi_{ij,2}$$

$$\tilde{a}_{\ell j}^{mi} = -2\int \mathrm{d}V(\xi)\, G\frac{\partial^2}{\partial \xi_j \partial \xi_\ell}\overline{v_i''v_m''(\xi)}$$

the problem now is to model the fourth-order tensor $\tilde{a}_{\ell j}^{mi}$.

The symmetry constraints of Ref. 2 hold, i.e.,

$$\tilde{a}_{\ell j}^{mi} = \tilde{a}_{\ell j}^{im} = \tilde{a}_{j\ell}^{im}$$

but the incompressibility condition does not, i.e.,

$$\tilde{a}_{ji}^{mi} \neq 0$$

in variable-density flows in general. The second-derivative term of the integrand of Eq. (10) becomes, upon contraction $\ell = i$ and use of Eq. (5),

$$\frac{\partial^2}{\partial x_i \partial y_\ell}\overline{v_m''(y)v_i''(x)} = \overline{\frac{\partial v_m''}{\partial y_\ell}\frac{\partial v_i''}{\partial x_i}} = -\frac{\partial\overline{v_m''v_i''}}{\partial y_\ell}\frac{\partial}{\partial x_i}\ell n\bar{\rho}$$

Again, if $\overline{v_m''(v)v_i''(x)}$ is nearly isotropic and if the correlation drops to zero rapidly enough compared with distance scales of significant variation of $\bar{\rho}$ and $\bar{\rho}\tilde{v}_\ell$, the first derivative will be odd in ξ_i. Therefore,

$$\tilde{a}_{\ell i}^{mi} \approx 0$$

This is not a consequence of incompressibility. It is caused by integrand behavior during integration.

Finally, it is still true, as it was in Ref. 2, that Green's theorem yields

$$\tilde{a}_{jj}^{im} = 2\overline{v_i''v_m''}$$

which leads to the identical conclusion of Ref. 2 insofar as modeling $\tilde{a}_{\ell j}^{mi}$. The result is

$$(\Phi_{ij} + \Phi_{ji})_2 = -\frac{(c_2+8)}{11}(P_{ij} - \tfrac{2}{3}P\delta_{ij}) - \frac{(30c_2-2)}{55}k\left(\frac{\partial\bar{\rho}\tilde{v}_i}{\partial x_j} + \frac{\partial\bar{\rho}\tilde{v}_j}{\partial x_i}\right) - \frac{(8c_2-2)}{11}(D_{ij} - \tfrac{2}{3}P\delta_{ij}) \quad (12)$$

where

$$P_{ij} = -\left(\overline{v_i''v_k''}\frac{\partial\bar{\rho}\tilde{v}_i}{\partial x_k} + \overline{v_j''v_k''}\frac{\partial\bar{\rho}\tilde{v}i}{\partial x_k}\right)$$

$$D_{ij} = -\left(\overline{v_i''v_k''}\frac{\partial\bar{\rho}\tilde{v}_k}{\partial x_j} + \overline{v_j''v_k''}\frac{\partial\bar{\rho}\tilde{v}_k}{\partial x_i}\right)$$

with $c_2 = 0.4$.

Equation (12) is the model proposed for the "rapid" part of the pressure-strain correlation. It has the following differences compared with the model of Jones:[1]

1) Density appears under the derivative.

2) The velocity fluctuation correlations appear with ordinary averages as opposed to Favre averages.

3) This model is redistributive without manipulation, i.e., $(\Phi_{ii} + \Phi_{ii})_2 = 0$, and the manipulation of Ref. 9 is not required or desired.

4) No added complication arises from mean dilation of the flow since density appears under the derivative and $\partial(\bar{\rho}\tilde{v}_i)/\partial x_i = 0$.

5) Because of 3), no assumption need be made concerning the smallness of $\overline{p'\partial v_\ell''/\partial x_\ell}$ in the overall velocity-pressure gradient correlation as has been made, e.g., by Bilger.[10] In fact, with this theory, it is zero.

Of course, Eq. (12) reduces to the constant-density result of Ref. 2 in the limit of constant density.

The reader should be aware that certain approximations have been selectively applied during the derivation and then not consistently applied. As an example, if isotropy of the velocity correlation tensors were consistently applied, the entire pressure-strain correlation would be zero. If $\rho'/\bar{\rho} \ll 1$ were consistently applied, $\tilde{v}_i \to \bar{v}_i$ and $v_j'' \to v_j'$; however, variable $\bar{\rho}$ would still remain under the derivative in Eq. (10), and this effect is the primary change from the behavior of Φ_{ij} as compared with that of Ref. 1. It appears to the author that Eq. (10) is the most natural form to evolve. Of course, it must be tested against experiment. This will be done, albeit inconclusively, later in this paper.

Now consider the pressure-scalar gradient correlation $\varphi_{j,2}$ of Eqs. (9). Forming the analog of Eqs. (10) and (11),

$$\varphi_{i,2} = \varphi_{i,2,I} + \varphi_{i,2,II}$$

$$\varphi_{i,2,I} = 2\int \mathrm{d}V(y)\, G\frac{\partial}{\partial y_m}(\bar{\rho}\tilde{v}_\ell)\frac{\partial^2}{\partial x_i \partial y_\ell}\overline{c''(x)v_m''(y)} \quad (13)$$

$$\varphi_{i,2,II} = 2\int \mathrm{d}V(y)\, G\bar{\rho}\tilde{v}_\ell\frac{\partial^2}{\partial x_i \partial y_m \partial y_\ell}\overline{c''(x)v_m''} \quad (14)$$

The triple derivative in the integrand of Eq. (14) may be written, with the use of Eq. (5), as

$$\frac{\partial^2}{\partial x_i \partial y_\ell}\left[\overline{c''(x)\frac{\partial v''_m}{\partial y_m}}\right] = -\frac{\partial^2}{\partial x_i \partial y_\ell}\left(\overline{c''v''_k}\frac{\partial \ell n \bar{\rho}}{\partial y_k}\right)$$

If $\overline{c''(x)v''_k(y)}$ is a nearly isotropic tensor[8]

$$\overline{c''v''_k} = \overline{c''v''(y)}K(r)\xi_k$$

and if $\partial(\ell n\bar{\rho})/\partial y_k$ is relatively stationary while the correlation tensor falls to zero, the second derivative $\partial^2/\partial x_i \partial y_\ell$ of the correlation tensor will be odd in ξ_k and will yield a null result in the integration process. This also requires that the correlation tensor fall to zero in a distance small compared with variation in G and $\bar{\rho}\tilde{v}_\ell$ in the integrand of Eq. (14). However, one is led to an inconsistency in the "quasi-isotropic" assumption. Note that in Eq. (13), after removing the mean derivative by the usual arguments, an isotropic tensor assumption would destroy this integral also. There is a difference here as compared with the pressure-strain correlation. The quasi-isotropy assumption does not produce different orders of magnitude between Eqs. (13) and (14) on the basis of quasi-isotropy, as it did in the examination of the two similar terms of the pressure-strain correlation.

The only way out of this dilemma appears to be to make a near-constant density assumption that $\ell \partial \ell n \bar{\rho}/\partial y_k$ is small compared with unity. Then $\varphi_{1,2,II} \ll \varphi_{i,2,I}$ and, by the usual arguments,

$$\varphi_{i,2} \approx \varphi_{i,2,II} \approx 2\frac{\partial}{\partial x_m}(\bar{\rho}\tilde{v}_\ell)\int \mathrm{d}V(y)\, G\frac{\partial^2}{\partial x_\ell \partial y_\ell}\overline{c''v''_m}$$

$$= \mathcal{B}^m_{i\ell}\frac{\partial}{\partial x_m}(\bar{\rho}\tilde{v}_\ell)$$

$$\mathcal{B}^m_{i\ell} = 2\int \mathrm{d}V(y)\, G\frac{\partial^2}{\partial x_i \partial y_\ell}\overline{c''v''_m} = -2\int \mathrm{d}V(\xi)\, G(\xi)\frac{\partial^2}{\partial \xi_i \partial \xi_\ell}\overline{c''v''_m} \tag{15}$$

The only symmetry applying to $\mathcal{B}^m_{i\ell}$ is $\mathcal{B}^m_{i\ell} = \mathcal{B}^m_{\ell i}$.

Moreover, by Green's theorem, $\mathcal{B}^m_{ij} = 2\overline{c''v''_i}$. The most general third-order tensor linear in the scalar transport correlations which satisfies the symmetry constraint is, in terms of two constants α and β,

$$\mathcal{B}^m_{i\ell} = \alpha\delta_{i\ell}\overline{c''v''_m} + \beta(\delta_{mi}\overline{c''v''_\ell} + \delta_{mi}\overline{c''v''_i})$$

In prior work,[3,7] the incompressibility condition $\mathcal{B}^m_{jm} = 0$ was employed; it is not true here, but has been used here in the form of a near-constant-density assumption. For in Eq. (15),

$$\frac{\partial^2}{\partial x_i \partial y_m}\overline{c''v''_m} = \overline{\frac{\partial c''}{\partial x_i}\frac{\partial v''_m}{\partial y_m}} = -\frac{\partial}{\partial x_i}\left(\overline{c''v''_k}\frac{\partial}{\partial y_k}\ell n\bar{\rho}\right)$$

Hence, one can apply the condition that $b^m_{\ell m}$ is very much less than all other $\mathcal{B}^m_{j\ell}$ and set it at zero. This, with the condition from Green's theorem, determines α and β uniquely with *no* undetermined constants. The result is similar to Jones' correlation suggestion but, just as with the pressure-strain correlation, there are important differences. The suggested correlation is

$$\varphi_{i,2} = -0.2(\delta_{j\ell}\overline{v''_i c''} + \delta_{i\ell}\overline{v''_j c''})\frac{\partial(\tilde{v}_j\bar{\rho})}{\partial x_\ell} + 0.8\overline{v''_j c''}\frac{\partial(\tilde{v}_j\bar{\rho})}{\partial x_j} \tag{16}$$

Equation (16) reduces to standard form in the truly constant-density limit. The major difference, when compared with Jones' suggestion, is the appearance of ρ under the derivative, just as with the pressure-strain correlation. However, this appearance is less satisfactory than before because a near-constant-density assumption had to be made. Nevertheless, Eq. (16) appears to be a more natural emerging form than other treatments.

III. Consequences and Comparison with Experiment

Experiments are underway at this laboratory to measure the pressure-strain and pressure-scalar gradient correlations in a premixed flame. Prior measurements have been obtained for the pressure-velocity correlation in such a flame,[11] but pressure-scalar, pressure-strain, and pressure-scalar gradient information has not yet been extracted. At the writing of this paper there are no direct tests of the theory presented here. All that may be checked is the consistency or inconsistency of partial indicators as to the correctness of the approach.

Hydrogen-Air Diffusion Flame

In Ref. 5, Jones' model was checked against indirect measurements of the pressure-strain correlation for jet flames of H_2 discharging into a coflowing stream of air. Let 1 denote the primary axial flow direction and 2 the direction transverse to 1. The jets tested have the property that $\partial/\partial x_1 \ll \partial/\partial x_2$ and $\tilde{v}_1 \gg \tilde{v}_2$. Moreover, in these jets, differences between Favre averages and ordinary averages have been found to be weak. Anisotropy has been found to be large, with $\widetilde{v''^2_1}$ often as large as twice $\widetilde{v''^2_2}$.

The only difference in the theory used for comparison and the one of this work lies in the "rapid" part of the pressure strain. After reduction, the only differences in the normal stress and shear stress equations are like

$$\bar{\rho}\widetilde{v''_1 v''_2}\frac{\partial \tilde{v}_1}{\partial x_2} \qquad \text{compared with} \qquad \overline{v''_1 v''_2}\frac{\partial \bar{\rho}\tilde{v}_1}{\partial x_2}$$

the latter forms coming from the theory here. It turns out, peculiar to this flow, that these two forms are not very different. Already mentioned is the weak difference between Favre and conventional averages. More importantly, for these flames, as one moves laterally from the axis, $\bar{\rho}$ is approximately constant due to the countering effects of molecular weight increase and temperature decrease. At a large enough lateral distance (the turbulent flame zone), the temperature will start to decrease if one proceeds further outward, and $\partial\bar{\rho}/\partial x_2$ can become large; however, at this point, $\tilde{v}_1$ is small. Consequently, this flow does not constitute a good test for comparison of theories. The results are the same regardless of the theory used. One positive finding, however, is that the new theory is *not inconsistent* with the experimental results.

It is also known[12] in these flames that transverse countergradient diffusion of mass, momentum, and energy is rare, if not nonexistent. Consider the scalar c to be the hydrogen element mass fraction, which in these flows is an approximately conserved scalar. Here, $\overline{v''_2 c''} > 0$ everywhere as the hydrogen diffuses outward by turbulent transport. In the scalar transport equation, positive φ_2 corresponds to an increase in transport and negative φ_2 corresponds to attempt to create countergradient diffusion. Looking at φ_2, the sign depends on the relative magnitude of $c\varphi_1$ and $c\varphi_2$, so the result is inconclusive. Noting the "rapid" part of Eq. (16), all terms are zero under the effective boundary-layer approximation. Consequently, there is no tendency toward countergradient diffusion through this effect.

Premixed Flames with No Streamline Deflection

If a premixed flame occurs in a confined duct, a reasonable first approximation might be that streamlines go through the flame zone undeflected from the freestream axial direction, regardless of flame orientation. A large body of theory has been based on such an approximation.[13] Ducted flames have also been experimentally studied, and the example used here is from Moreau,[14] who found the axial component of turbulent intensity to markedly increase in passing through a flame zone inclined to the freestream direction. In an undeflected streamline flow, the following properties hold:

$$\frac{\partial \bar{\rho}\tilde{v}_1}{\partial x_1}=0, \quad \frac{\partial \bar{p}}{\partial x_1}<0, \quad \tilde{v}_2=0, \quad \frac{\partial \bar{\rho}\tilde{v}_1}{\partial x_2}=0, \quad \frac{\partial \tilde{v}_1}{\partial x_2}<0$$

for flames inclined toward the downstream direction where a two-dimensional planar flow is assumed.

Looking only at the "rapid" parts of the pressure-strain and pressure-scalar gradient correlations, it is first noted that the current theory yields $(\Phi_{11}+\Phi_{11})_2=(\Phi_{22}+\Phi_{22})_2=\varphi_{1,2}=0$. It is to be noted that this is *assumed* in Ref. 13 but it is a theoretical consequence here. On the other hand, if ρ does not appear under the derivative, as in Ref. 1,

$$(\Phi_{11}+\Phi_{11})_2<0, \qquad (\Phi_{22}+\Phi_{22})_2>0$$

The second consequence cannot be judged, but the first says that the pressure-strain correlation acts as a sink for the axial component of turbulence. This is not proof, of course, that the current theory is superior because there are other effects, but it appears to indicate that the current theory may be more accurate than others. Concerning the pressure-scalar gradient term, it is nonzero if ρ is out from under the derivative, but there is no experimental guidance here. It should also be mentioned that there is some danger in attempting to use the current theory in confined flows without making a near-wall correction, as in Ref. 2.

Premixed Jet Flames

A large body of data[11] exists on measurement of the pressure-velocity (not pressure-strain) correlation on an axially symmetric premixed propane-air turbulent jet flame. The conclusions were that 1) the pressure-velocity correlation can be either a sink or a source for turbulent kinetic energy, depending upon measurement position, 2) the correlation is a source of axial normal stress in the flame zone on the flow axis, 3) countergradient as well as usual gradient scalar diffusion can take place in the axial direction along the flame axis, 4) there was a mild but monotonic increase in the turbulence, primarily in the lateral direction (toward isotropy), in passing through the flame zone, and 5) the mean axial velocity on the axis was nearly invariant in passing through the flame zone. That is, on the axis,

$$\frac{\partial \tilde{v}_1}{\partial x_1}\approx 0, \qquad \frac{\partial \tilde{v}_1}{\partial x_2}=0, \qquad \tilde{v}_2=0, \qquad \frac{\partial \bar{p}}{\partial x_1}<0$$

$$\overline{v_1''c''} \begin{cases} <0 & \text{early in the flame} \\ >0 & \text{later in the flame} \end{cases}$$

where c is temperature.

Now, if Ref. 1 is used, $(\Phi_{11}+\Phi_{11})_2=(\Phi_{22}+\Phi 22)_2=\varphi_{1,2}=0$ because all mean velocity gradients on the axis are small. For this reason there is no turbulence production, only the effects of dissipation, diffusion, and pressure. But the first term on the right side of Eq. (1) is small because in this unbounded flow the mean pressure gradient is small. The known pressure-velocity correlation could account for the rise in k along the axis, and the "return to isotropy" term could account for the observed tendency of $\overline{v_2''^2}$. But the problem is that $\overline{v_1''^2}$ remains nearly stationary in passing through the flame, so there must be a sink in the balance equation for $\overline{v_1''^2}$. This can only come from either diffusion, which was not measured, or the pressure-strain correlation.

Using the theory here and working out the details,

$$(\Phi_{11}+\Phi_{11})_2\approx 0.21k\frac{\partial \bar{\rho}\tilde{v}_1}{\partial x_1}<0$$

$$(\Phi_{22}+\Phi_{22})_2\approx -0.58\overline{v_1''^2}\frac{\partial \bar{\rho}\tilde{v}_1}{\partial x_1}>0$$

which is precisely the behavior required. This is considered the most positive test of the theory; however, it is demonstrated without knowing diffusion effects.

Another test comes from calculations of

$$\varphi_{1,2}=0.4\overline{v_1''c''}\frac{\partial \bar{\rho}\tilde{v}_1}{\partial x_1}$$

Early in the flame, countergradient diffusion was noted so that $\overline{v_1''c''}<0$, and this gives way to a positive value as one passes through the flame. At least $\varphi_{1,2}$ can give such behavior. But if density were not under the derivative, $\varphi_{1,2}$ would be zero.

In summary, considering the scant available experimentation, no situation has been found where the theory of this paper should be discarded compared with the suggestion of Ref. 1. Indeed, there is some evidence that the current proposal should be provisionally accepted.

IV. Conclusions

1) Using various approximations of quasi-isotropy, near homogeneity, weak variation of mean quantities, and sufficiently small density fluctuations, new forms are proposed for the pressure-strain and pressure-scalar gradient correlations in variable-density turbulent reacting flows.

2) When compared against scant available experimental data, these proposed correlations are not inconsistent with the data, and, in one case, appear to be required for agreement with the data.

3) The proposed correlations collapse to accepted incompressible forms in the limit of constant density.

4) The pressure-strain correlation as developed here is redistributive, obviating the need for a commonly made assumption rather than a theoretical consequence.

5) The models suggest that common neglect of pressure-strain and pressure-scalar gradient correlations is inappropriate, in general.

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